Solution to HW 6

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MATH 2020 B HW 6

Due Date: Mar 20, 2020 (12:00 noon)

Thomas' Calculus (12th Ed.)

§16.1 : 14, 17, 20, 23

§16.2 : 10, 15, 20, 31, 45, 53

§16.1

Evaluating Line Integrals over Space Curves

14. Find the line integral of $f(x, y, z) = \sqrt{3}/(x^2 + y^2 + z^2)$ over the curve $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $1 \le t \le \infty$.

Sol) $\vec{r}(t) = t\vec{i} + t\vec{j} + t\vec{k}$; $\vec{r}'(t) = \vec{i} + \vec{j} + \vec{k}$; $|\vec{r}'(t)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

$f(x,y,z) = \frac{\sqrt{3}}{x^2+y^2+z^2}$; $f(\vec{r}(t)) = \frac{\sqrt{3}}{t^2+t^2+t^2} = \frac{\sqrt{3}}{3} \cdot \frac{1}{t^2}$.

$\therefore \int_C f(x,y,z)\,ds = \int_1^\infty \left(\frac{\sqrt{3}}{3} \cdot \frac{1}{t^2}\right)(\sqrt{3}\,dt) = \int_1^\infty \frac{1}{t^2}\,dt = \left(\lim_{t\to+\infty}\left(-\frac{1}{t}\right)\right) - \left(-\frac{1}{t}\right)\Big|_{t=1} = 1$ //

17. Integrate $f(x, y, z) = (x + y + z)/(x^2 + y^2 + z^2)$ over the path $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 < a \le t \le b$.

Sol) $\vec{r}(t) = t\vec{i} + t\vec{j} + t\vec{k}$; $\vec{r}'(t) = \vec{i} + \vec{j} + \vec{k}$; $|\vec{r}'(t)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

$f(x,y,z) = \frac{x+y+z}{x^2+y^2+z^2}$; $f(\vec{r}(t)) = \frac{t+t+t}{t^2+t^2+t^2} = \frac{1}{t}$.

$\therefore \int_C f(x,y,z)\,ds = \int_a^b \left(\frac{1}{t}\right)(\sqrt{3}\,dt) = \sqrt{3}\int_a^b \frac{1}{t}\,dt = \sqrt{3}\,[\ln|t|]_a^b = \sqrt{3}\,(\ln|b| - \ln|a|) = \sqrt{3}\ln\frac{b}{a}$ //

(since $a, b > 0$)

Line Integrals over Plane Curves

20. Evaluate $\int_C \sqrt{x + 2y}\, ds$, where C is

a. the straight-line segment $x = t$, $y = 4t$, from $(0, 0)$ to $(1, 4)$.

b. $C_1 \cup C_2$; C_1 is the line segment from $(0, 0)$ to $(1, 0)$ and C_2 is the line segment from $(1, 0)$ to $(1, 2)$.

Sol) (a) $\vec{r}(t) = t\vec{i} + 4t\vec{j}$, when $0 \leq t \leq 1$; $\vec{r}'(t) = \vec{i} + 4\vec{j}$; $|\vec{r}'(t)| = \sqrt{1^2 + 4^2} = \sqrt{17}$.

$f(x,y) = \sqrt{x+2y}$; $f(\vec{r}(t)) = \sqrt{t+8t} = 3\sqrt{t}$

$\therefore \int_C f(x,y)ds = \int_0^1 (3\sqrt{t})(\sqrt{17}dt) = 3\sqrt{17}\int_0^1 \sqrt{t}\, dt = 3\sqrt{17}\left[\frac{t^{3/2}}{3/2}\right]_0^1 = 2\sqrt{17}$ //

(b) $C_1 : \vec{r}_1(t) = t\vec{i}$, when $0 \leq t \leq 1$; $\vec{r}_1'(t) = \vec{i}$; $|\vec{r}_1'(t)| = \sqrt{1^2} = 1$.

$C_2 : \vec{r}_2(t) = \vec{i} + t\vec{j}$, when $0 \leq t \leq 2$; $\vec{r}_2'(t) = \vec{j}$; $|\vec{r}_2'(t)| = \sqrt{1^2} = 1$.

$f(x,y) = \sqrt{x+2y}$; $f(\vec{r}_1(t)) = \sqrt{t}$; $f(\vec{r}_2(t)) = \sqrt{1+2t}$

$\therefore \int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds = \int_0^1 (\sqrt{t})(dt) + \int_0^2 (\sqrt{1+2t})(dt)$

$= \left[\frac{t^{3/2}}{3/2}\right]_0^1 + \left[\frac{1}{2}\frac{(1+2t)^{3/2}}{3/2}\right]_0^2 = \frac{2}{3} + \frac{1}{3}(5\sqrt{5} - 1) = \frac{1+5\sqrt{5}}{3}$ //

23. Evaluate $\int_C \frac{x^2}{y^{4/3}}\, ds$, where C is the curve $x = t^2, y = t^3$, for $1 \leq t \leq 2$.

Sol) $\vec{r}(t) = t^2\vec{i} + t^3\vec{j}$; $\vec{r}'(t) = 2t\vec{i} + 3t^2\vec{j}$; $|\vec{r}'(t)| = \sqrt{(2t)^2 + (3t^2)^2} = t\sqrt{4+9t^2}$.

$f(x,y) = \frac{x^2}{y^{4/3}}$; $f(\vec{r}(t)) = \frac{(t^2)^2}{(t^3)^{4/3}} = 1$.

$\therefore \int_C f(x,y)ds = \int_1^2 (1)(t\sqrt{4+9t^2}dt) = \left[\frac{1}{18}\cdot\frac{(4+9t^2)^{\frac{3}{2}}}{3/2}\right]_1^2 = \frac{1}{27}\left[80\sqrt{10} - 13\sqrt{13}\right]$ //

§16.2

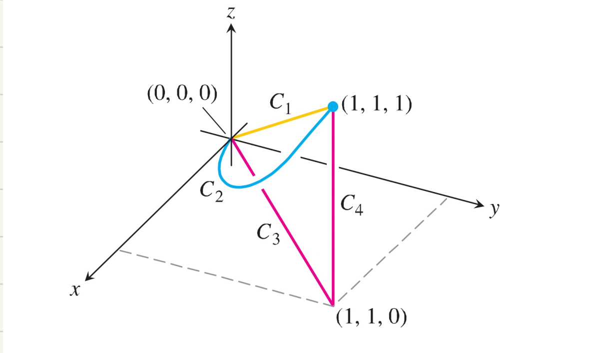
Line Integrals of Vector Fields

In Exercises 7–12, find the line integrals of $\mathbf{F}$ from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths in the accompanying figure.

a. The straight-line path C_1: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$

b. The curved path C_2: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}, \quad 0 \le t \le 1$

c. The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the segment from $(1, 1, 0)$ to $(1, 1, 1)$



10. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

Sol) (a) C_1: $\vec{r}(t) = t\vec{i} + t\vec{j} + t\vec{k}$;

$\vec{r}'(t) = \vec{i} + \vec{j} + \vec{k}$.

$\vec{F}(\vec{r}(t)) = t^2\vec{i} + t^2\vec{j} + t^2\vec{k}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = t^2 + t^2 + t^2 = 3t^2$

$\therefore \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 3t^2 dt = [t^3]_0^1 = 1$ //

(b) C_2: $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^4\vec{k}$; $\vec{r}'(t) = \vec{i} + 2t\vec{j} + 4t^3\vec{k}$.

$\vec{F}(\vec{r}(t)) = t^3\vec{i} + t^6\vec{j} + t^5\vec{k}$; $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = t^3 + 2t^7 + 4t^8$.

$\therefore \int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 (t^3 + 2t^7 + 4t^8) dt = \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4t^9}{9}\right]_0^1 = \frac{1}{4} + \frac{1}{4} + \frac{4}{9} = \frac{17}{18}$ //

(c) C_3: $\vec{r}_3(t) = t\vec{i} + t\vec{j}$, where $0 \le t \le 1$; $\vec{r}_3'(t) = \vec{i} + \vec{j}$. $\vec{F}(\vec{r}_3(t)) = t^2\vec{i}$; $\vec{F}(\vec{r}_3(t)) \cdot \vec{r}_3'(t) = t^2$.

C_4: $\vec{r}_4(t) = \vec{i} + \vec{j} + t\vec{k}$, where $0 \le t \le 1$; $\vec{r}_4'(t) = \vec{k}$.

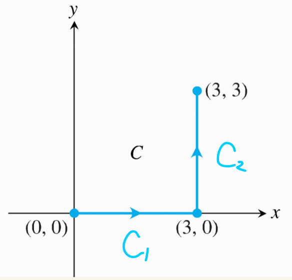
$\vec{F}(\vec{r}_4(t)) = \vec{i} + t\vec{j} + t\vec{k}$; $\vec{F}(\vec{r}_4(t)) \cdot \vec{r}_4'(t) = t$

$\therefore \int_{C_3 \cup C_4} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r}_3 + \int_{C_4} \vec{F} \cdot d\vec{r}_4 = \int_0^1 t^2 dt + \int_0^1 t\, dt = \left[\frac{t^3}{3}\right]_0^1 + \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ //

Line Integrals with Respect to *x*, *y*, and *z*

In Exercises 13–16, find the line integrals along the given path C.

15. $\int_C (x^2 + y^2)\, dy$, where C is given in the accompanying figure.



Sol) $C_1: \vec{r}_1(t) = t\vec{i}$, where $0 \le t \le 3$; $dy = 0$

$C_2: \vec{r}_2(t) = 3\vec{i} + t\vec{j}$, where $0 \le t \le 3$; $dy = dt$.

$f(x,y) = x^2 + y^2$; $f(\vec{r}_2(t)) = 9 + t^2$.

$$\therefore \int_C f(x,y)\,dy = \int_{C_1} f(x,y)\,dy + \int_{C_2} f(x,y)\,dy = 0 + \int_0^3 (9+t^2)\,dt = \left[9t + \frac{t^3}{3}\right]_0^3 = 36 \;//$$

Work

In Exercises 19–22, find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.

20. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$
$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \le t \le 2\pi$

Sol) $C: \vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + \left(\frac{t}{6}\right)\vec{k}$, where $0 \le t \le 2\pi$; $\vec{r}'(t) = -(\sin t)\vec{i} + (\cos t)\vec{j} + \frac{1}{6}\vec{k}$.

$\vec{F}(\vec{r}(t)) = (2\sin t)\vec{i} + (3\cos t)\vec{j} + (\cos t + \sin t)\vec{k}$;

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = -2\sin^2 t + 3\cos^2 t + \frac{1}{6}(\cos t + \sin t)$.

$$\therefore \text{Work done by } \vec{F} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(-2\sin^2 t + 3\cos^2 t + \frac{1}{6}(\cos t + \sin t)\right)dt$$

$$= \left[-2\left(\frac{t}{2} - \frac{\sin 2t}{4}\right) + 3\left(\frac{t}{2} + \frac{\sin 2t}{4}\right) + \frac{1}{6}(\sin t - \cos t)\right]_0^{2\pi} = \pi \;//$$

Work, Circulation, and Flux in the Plane

In Exercises 31–34, find the circulation and flux of the field **F** around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le \pi$, followed by the line segment $\mathbf{r}_2(t) = t\mathbf{i},\ -a \le t \le a$.

31. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

Sol) $\vec{r}_1(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j}$; $\vec{r}_1{}'(t) = (-a\sin t)\vec{i} + (a\cos t)\vec{j}$;

$\vec{F}(\vec{r}_1(t)) = (a\cos t)\vec{i} + (a\sin t)\vec{j}$: $\vec{F}(\vec{r}_1(t))\cdot\vec{r}_1{}'(t) = -a^2\cos t\sin t + a^2\sin t\cos t = 0$

$\therefore$ Circulation along $\vec{r}_1 = 0$

Flux along $\vec{r}_1 = \int_0^{\pi}\big((a\cos t)(a\cos t) - (a\sin t)(-a\sin t)\big)dt = \int_0^{\pi} a^2\,dt = \pi a^2$

$\vec{r}_2(t) = t\vec{i}$; $\vec{r}_2{}'(t) = \vec{i}$

$\vec{F}(\vec{r}_2(t)) = t\vec{i}$: $\vec{F}(\vec{r}_2(t))\cdot\vec{r}_2{}'(t) = t$

$\therefore$ Circulation along $\vec{r}_2 = \int_{-a}^{a} t\,dt = \left[\frac{t^2}{2}\right]_{-a}^{a} = 0$

Flux along $\vec{r}_2 = \int_{-a}^{a}\big((t)(0) - (0)(1)\big)dt = 0$

$\therefore$ Total circulation $= 0 + 0 = 0$ //

Total Flux $= \pi a^2 + 0 = \pi a^2$ //

Vector Fields in the Plane

45. Work and area Suppose that $f(t)$ is differentiable and positive for $a \le t \le b$. Let C be the path $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}$, $a \le t \le b$, and $\mathbf{F} = y\mathbf{i}$. Is there any relation between the value of the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and the area of the region bounded by the t-axis, the graph of f, and the lines $t = a$ and $t = b$? Give reasons for your answer.

Sol) Yes by the following claim:

Claim $\int_C \vec{F} \cdot d\vec{r} = \int_a^b f(t)\,dt$ (= area in question, since $f(t)$ is positive for any $a \le t \le b$)

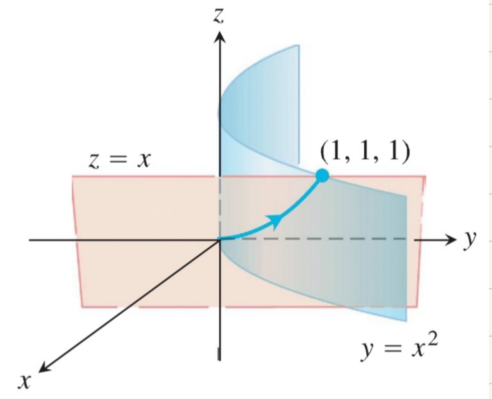
Proof) C : $\vec{r}(t) = t\vec{i} + f(t)\vec{j}$; $\vec{r}'(t) = \vec{i} + f'(t)\vec{j}$.

$\vec{F}(\vec{r}(t)) = f(t)\vec{i}$; $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = f(t)$

$\therefore$ LHS $= \int_C \vec{F} \cdot d\vec{r} = \int_a^b f(t)\,dt =$ RHS.

Flow Integrals in Space

53. Flow along a curve The field $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ is the velocity field of a flow in space. Find the flow from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (*Hint:* Use $t = x$ as the parameter.)



Sol) Let $x = t$; then $y = t^2$; $z = t$

$\therefore$ The curve of intersection is given by

$\vec{r}(t) = t\vec{i} + t^2\vec{j} + t\vec{k}$, where $0 \le t \le 1$.

$\vec{r}'(t) = \vec{i} + 2t\vec{j} + \vec{k}$; $\vec{F}(\vec{r}(t)) = t^3\vec{i} + t^2\vec{j} - t^3\vec{k}$; $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = t^3 + 2t^3 - t^3 = 2t^3$.

$\therefore$ Flow $= \int_C \vec{F} \cdot d\vec{r} = \int_0^1 2t^3\,dt = \left[\frac{t^4}{2}\right]_0^1 = \frac{1}{2}$ //